PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On the Stress-Optical Effect in Transparent Solids Strained beyond the Elastic Limit

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III. *On the Stress-optical Effect in Transparent Solids strained beyond the Elastic Limit.*

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§ 1. *Introduction and References.*

Experiments carried out by one of the present authors with a slab of glass under flexure have indicated that artificial double refraction in glass strained beyond the elastic limit is probably proportional to the stress rather than to the strain (*cf.* L. N. G. Filon, 'Phil. Trans.,' A, vol. 207, pp. 303–305). More recently the experiments of Prof. E. G. Coker and Mr. K. C. Chakko ("The Stress-Strain Properties of Nitro-Cellulose and the Law of its Optical Behaviour," 'Phil. Trans.,' A, vol. 221, pp. 139–162) have suggested that in celluloid or xylonite, whilst for the highest loads the double refraction is no longer proportional to either the stress or the strain, it is more nearly proportional to the former than to the latter.

Further, casual observations in the past had shown one of us that, when a piece of transparent material, whether glass or celluloid, which had been overstrained was released, a certain amount of residual illumination was visible between crossed nicols in certain cases. This illumination gradually died out, showing that the artificial double refraction exhibited some of the characteristics of permanent set, with a slow recovery. No precise measurements of this effect, however, seem ever to have been made.

In the few isolated cases observed by one of us, the stress distribution was a complex one. In such a case, when there is overstrain and the load is removed, the less strained portions will in general react upon the permanently overstrained ones, and thus an internal stress system is introduced, which produces its own optical effect; but this optical effect is the consequence of a mechanical readjustment and is not a true optical residual, that is, a double refraction persisting after the stress has been entirely removed.

It seemed, therefore, of interest to examine whether in fact such optical permanent set did exist, as if so, it would apparently settle the question whether stress or strain is the immediate cause of the phenomenon of double refraction.

In addition, this opened up the whole question of time effects in this connection, of which apparently no observations had been made to date, and it was hoped that a study of such effects might throw some light upon the mechanism of photo-elasticity, of which very little is really known.

The plan adopted was to observe a test-piece under simple stress (pressure in the case of glass and tension in that of xylonite or celluloid) between crossed nicols. Sodium light was employed throughout, and changes of relative retardation were measured by the shift of the black band produced by a Babinet compensator.

In such a case of simple loading the applied stress should disappear throughout immediately the load is removed, *provided the material is homogeneous.* The importance of this last restriction will appear in § 13.

As the experiments progressed new lines of inquiry suggested themselves, so that, ultimately, the whole subject of time effect or creep, for strain as well as for double refraction, came under consideration, more particularly in xylonite.

§ 2. *Apparatus and Method of Observation.*

The apparatus used for applying the compression stress to glass blocks was substantially that described by one of us in 'Camb. Phil. Soc. Proc.,' vol. 12, part 1, p. 56. The specimen S was placed between two knife-edge blocks B, B (fig. 1), one of the knife-

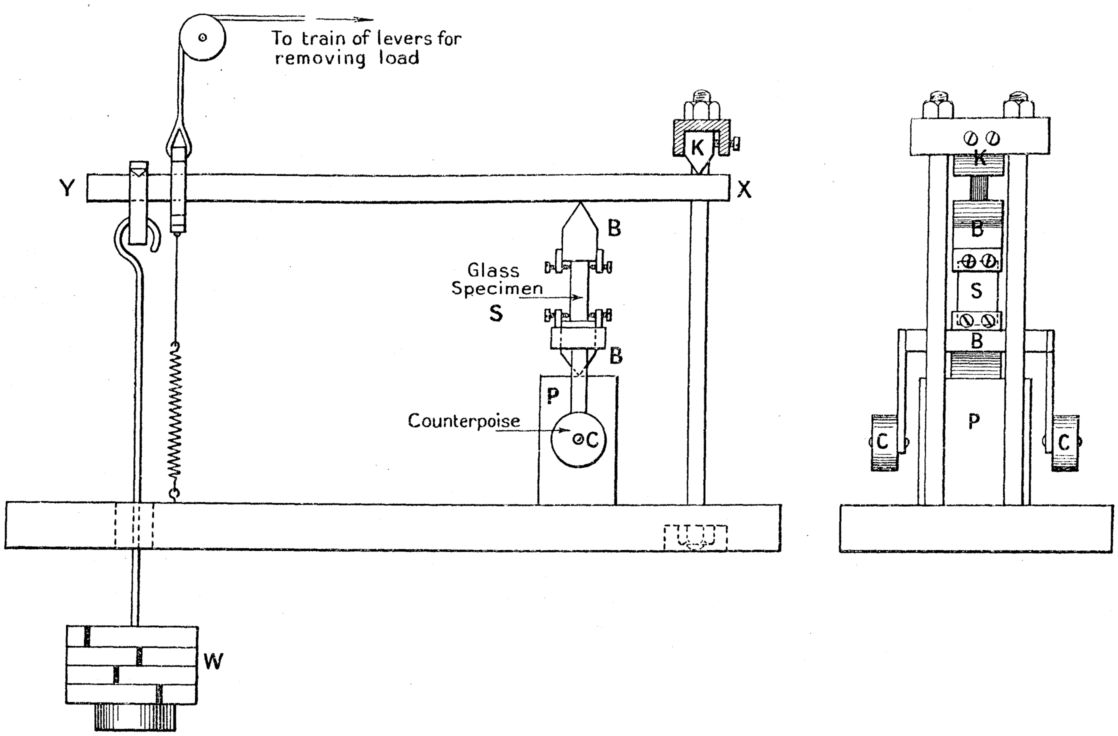

Fig. 1.

edge blocks resting on a solid support P and the other being pressed down by a lever XY, one end of which X bore against a fixed knife-edge K, the other end carrying the weight W. In order that the block and knife-edges might remain in position when the whole weight was removed, a counterpoise C was attached to the lower knife-edge, and this maintained the balance of the whole when the lever was lifted.

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The centering of the specimen was effected by means of adjusting screws passing through a metal frame attached to each knife-edge block, as shown in fig. 1. In order to increase the field, as large nicols were not available, the polarizer was replaced by a pile of plates inclined so as to polarize the light at 45° to the line of stress.

The compensator consisted of two quartz wedges, each 1 inch long and each cut so as to give a relative retardation of λ at one end and $\frac{1}{4}\lambda$ at the other. Thus the two, superposed, gave a range of $\frac{3}{4}\lambda$ to $-\frac{3}{4}\lambda$. Viewed between the crossed polarizer and analysing nicol, this showed a bright field crossed by a fairly sharply defined black band at the midpoint, where the relative retardation produced by the wedges was zero.

When the load was applied this black band moved up or down to a point on the wedge where the relative retardation was equal in magnitude but opposite in sign to that produced in the specimen. As the load on the glass was increased, this zero retardation band moved out of the field of view and was followed by other bands corresponding to retardations of λ, 2λ, 3λ, &c., respectively.

In order to measure the shift of these bands it was necessary to devise a scale which should be transparent between crossed nicols. A scale was accordingly cut on a sheet of transparent celluloid, which was attached under slight strain to the compensator holder. Test measurements were made at intervals to ensure that the scale had not altered under the strain.

It was found that by registering the band on a movable cross-wire, the shift could be read to an accuracy of 0·01 of an inch, corresponding to a relative retardation of 0·015 of a wave-length of sodium light.

The uniformity of the stress in the specimen could be very accurately tested by means of the Babinet compensator, as a slight error in centering of the block produced a decided obliquity in the black band when a large load was applied.

§ 3. *Experiments on Glass.*

A number of blocks of glass, 3 cm. × 4 cm. × 1 cm., were tested, the pressure being applied to the ends 3 cm. × 1 cm., and the light travelling parallel to the edge of length 3 cm.

TABLE of Glasses Tested.

Ref. No.	Makers' identification No.	μ_D.	Dispersion C to F.	Specific gravity.	General chemical composition.	Per cent. of PbO.
1	O 152	1·5368	0·01049	2·76	PbO, K_2O, Na_2O, (CaO), SiO_2 . . .	16
2	O 154	1·5710	0·01327	3·16	PbO, K_2O, Na_2O, (ZnO), (BaO), SiO_2	31
3	O 103	1·6202	0·01709	3·63	PbO, K_2O, SiO_2	47
4	O 192	1·6734	0·02104	4·10	PbO, K_2O, SiO_2	56
5	O 41	1·7174	0·02434	4·49	PbO, K_2O, SiO_2	63
6	4840	1·7537	0·02743	4·78	Unknown	—

(Substances in brackets occur in very small quantities only.)

Glasses 1 to 5 were provided and cut by Messrs. Zeiss before the war, and are described by their catalogue numbers.

Glass 6 was from Messrs. Hilger, and the number shown is the number of the melting.

These glasses were subjected to loads as under :—

Glass.	Pressure kg. wt./cm.^{2}	Relative retardation in wave-lengths.	Time load was left on in hours.
O 152	198	3	24
O 154	194	3	24
O 103	212	3	67
O 192	175	2	17
O 41	220	2	40
4840	149	1	24

In no case was there any increase in the observed retardation during the time the load was left on, nor was there any trace of residual retardation on removal of the load.

The apparatus employed allowed of the load being put on and removed very rapidly and smoothly, the effect being observed at the same time ; but in every case the band appeared to take up instantaneously its final position. Had there been a creep or a residual retardation as great as $0 \cdot 01 \lambda$ it must have been observed.

The loads applied approached in most cases as near to the breaking load as we dared go. Experiments, described elsewhere by one of us (H. T. Jessop, 'Phil. Mag.,' vol. xlii, Oct., 1921), indicate that plastic flow under pressure sets in, for glass, at much lesser stresses.

So far, then, as these experiments go, there is no indication of any residual stress-optical effect, or that the artificial double refraction in glass at ordinary temperatures is dependent upon any other factor than the stress actually applied. An attempt was made later to measure directly the strain in the glass, but the experimental difficulties are great and have not yet been overcome.

§ 4. *Methods of Observation of Celluloid Specimens.*

The next step was to examine celluloid, where residual effects previously observed had been much more marked. Here several difficulties were encountered.

In the first place it was found impossible to obtain celluloid without considerable and very irregular initial stress, and very marked "skin" effects were produced on cutting a specimen. Attempts were made to anneal the celluloid by heating it for several hours in a sealed vessel enclosed in a steam jacket, and allowing it to cool very slowly. This made a considerable improvement in the uniformity of the specimen under polarized light, but the warping produced necessitated reshaping after annealing, which led to the reintroduction of the "skin" stresses.

On account of this and of the unknown change of composition due to evaporation of camphor during the process, it was thought better to give up all attempts at annealing, and merely to select the specimens which showed greatest uniformity in their initial stresses.

In the second place it was impossible to use celluloid in large thicknesses without impairing its optical efficiency to a prohibitive extent. It had thus to be employed in

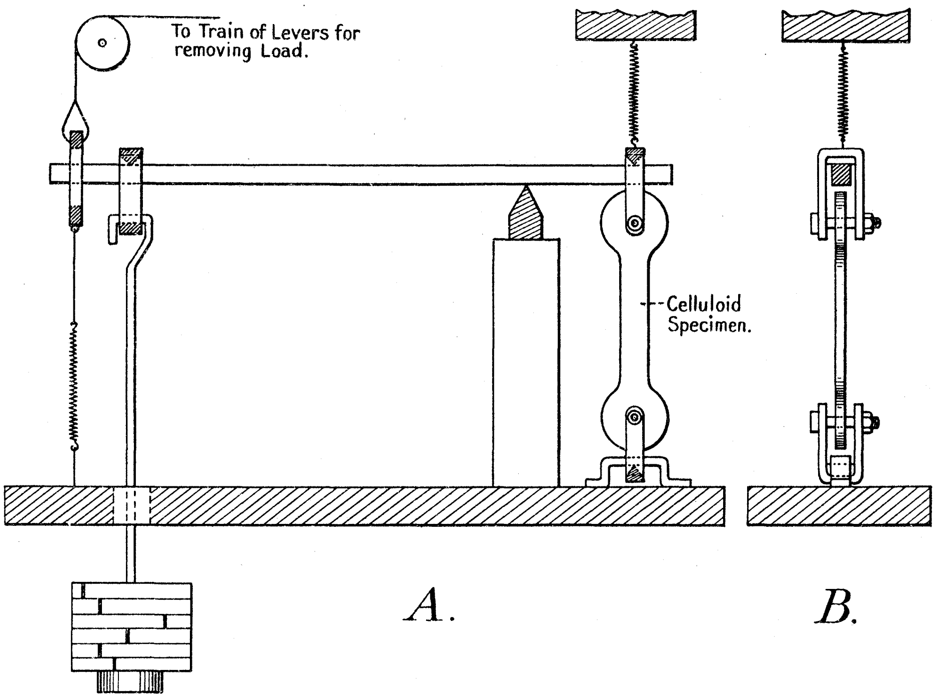

Fig. 2.

comparatively thin slabs and, owing to its low rigidity, buckling under pressure could not be avoided without unduly reducing the size of the block. Thus the type of stress examined was restricted to simple tension, the straining apparatus being as shown in fig. 2, and the specimens pierced with holes at the two ends, to enable the pull to be applied.

In the case of celluloid, it was found that the dark band, instead of being straight, was irregular, and that lack of uniformity in the material caused the shape of the band to alter from point to point of the specimen. It was accordingly necessary to observe always the same point of the specimen. The cross-wire was therefore kept in a fixed position on a stand, and the compensator fitted into a vertical slide and moved up and down by means of a micrometer screw and a spring. The readings of the micrometer screw then gave the retardation on a determinate scale.

Fig. 3 shows a typical example of the appearance of the band in a celluloid strip, and fig. 4 shows the modified apparatus for measuring the stress-optical effect.

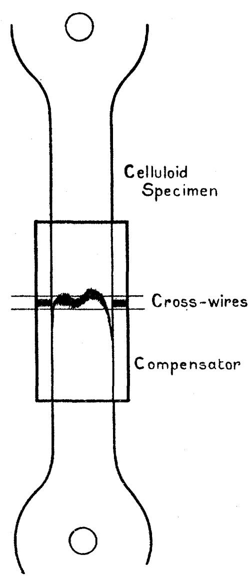

Fig. 3.

The shape of any one band did not alter appreciably during the time that a set of observations was being taken, but the bands for zero, λ, 2λ, 3λ, &c., retardations were not quite alike, so that small errors probably occurred in the *total* retardations measured, though these would not affect the time-creep.

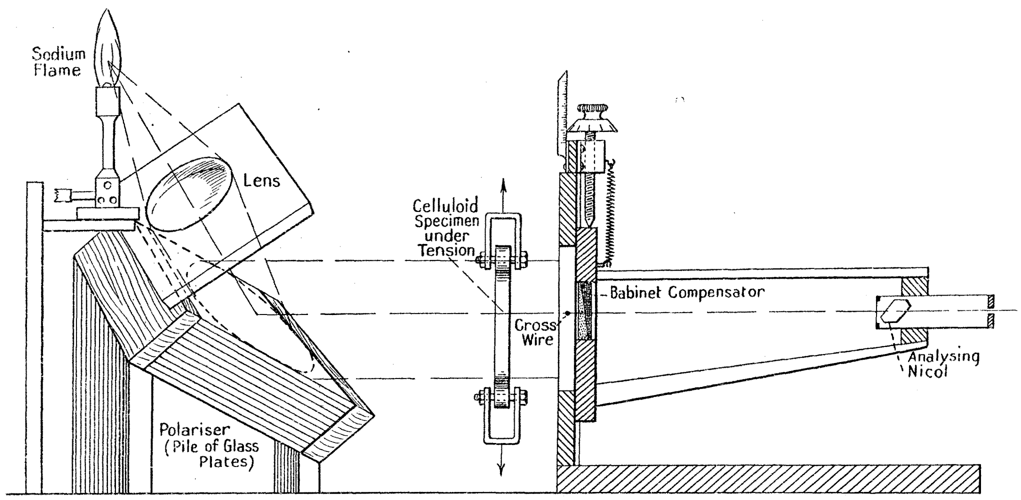

Fig. 4.

§ 5. *General Description of Optical Creep in Celluloid.*

On trying the very first experiment with celluloid, two facts immediately became apparent.

(1) After applying the load the retardation at once attained a certain initial value, which, however, was very difficult to determine directly with precision, as it was impossible to apply the load suddenly enough without starting vibrations and, in addition, to observe the band at the same instant.

The retardation, however, increased with the time, the load now remaining constant throughout. This increase was rapid at first, but its rate steadily diminished. It continued, for the higher and medium loads, for hours, and indeed, in some cases, there were still traces of creep when the load had been kept on several days.

This gradual increase of the retardation we shall refer to as the optical creep on loading.

(2) On removing the load the retardation did not immediately disappear, a quite considerable residual retardation remaining, which varied with the load applied and with the time this load had been kept on. This residual retardation was difficult to measure exactly, for reasons similar to those which affected the determination of the initial retardation on loading.

As time went on this residual retardation gradually diminished, rapidly at first, then more slowly. This process we shall speak of as the optical creep on recovery. The optical creep on recovery eventually continued until the residual retardation entirely disappeared. In order to make quite sure that the specimen had properly recovered optically, successive experiments with the same specimen were usually separated by at least a fortnight.

§ 6. *First Provisional Theory of Strain and Optical Creep.*

Curves of optical creep on loading and recovery plotted to time having been obtained, it was at first attempted to fit an exponential curve to them; this appearing probable on the following grounds.

It was assumed that the stress in the material was made up of two parts, one elastic and one viscous. Denoting by s_1 and s_2 the longitudinal and transverse stretches under a total tension T we then find

$$\mathrm{T} = \lambda(s_1+2s_2)-p+2(\mu+\nu\,\partial/\partial t)\,s_1, \quad (1)$$

$$0 = \lambda(s_1+2s_2)-p+2(\mu+\nu\,\partial/\partial t)\,s_2, \quad (2)$$

where λ, μ are the elastic constants of Lamé, ν is Stokes' coefficient of viscosity and

$$p-\tfrac{2}{3}\nu(\partial/\partial t)(s_1+2s_2) \quad (3)$$

is the so-called "hydrostatic pressure" in the viscous fluid.

Denote the cubical dilatation s_1+2s_2 by δ, we find from (1) and (2)

$$\mathrm{T} = (3\lambda+2\mu+2\nu\,\partial/\partial t)\,\delta-3p. \quad (4)$$

It is clear that we can get no farther towards a solution without making some assumption concerning the value of p. The most reasonable assumption appeared to be that the "hydrostatic pressure" was proportional to the total stress applied, or

$$p = \tfrac{2}{3}\nu\,\partial\delta/\partial t+\gamma\mathrm{T}. \quad (5)$$

Substituting this into (4), we find

$$\mathrm{T} = (3\lambda+2\mu)\,\delta/(1+3\gamma). \quad (6)$$

Substituting now into (1), after some reductions

$$(\mu+\nu\,\partial/\partial t)\{s_1-\tfrac{1}{3}(1+3\gamma)\,\mathrm{T}/(3\lambda+2\mu)\} = \tfrac{1}{3}\mathrm{T}, \quad (7)$$

which gives the differential equation satisfied by s_1, on this hypothesis.

The complete solution of equation (7) is well known, and can be expressed in the form

$$s = \tfrac{1}{3}(1+3\gamma)\,\mathrm{T}/(3\lambda+2\mu)+(1/3\nu)\,e^{-\mu t/\nu}\int_0^t e^{\mu t/\nu}\mathrm{T}\,dt, \quad (8)$$

T being a given function of the time which starts from zero at $t=0$, when $s=0$.

In view of experiments (to be described later) on the actual extension of the celluloid specimen, it will be convenient to interpret this solution when a tension T_0 is put on suddenly at $t=0$, maintained constant until $t=t_\omega$ and then removed.

Since the integrand in (8) is always finite and *bounded* (for, even when vibrations are taken into account, the tension on suddenly applying the load can at most reach the approximate value $2T_0$, as is well known) the integral in (8) cannot introduce a discontinuity in s.

The first term introduces a discontinuity, so that we have an *initial strain on loading* given by

$$s_0 = \tfrac{1}{3}(1+3\gamma)\,T_0/(3\lambda+2\mu) = \tfrac{1}{3}(1-2\eta)(1+3\gamma)\,T_0/E,$$

where η is POISSON's ratio and E is YOUNG's modulus. Note carefully that this initial strain is proportional to the load applied.

The strain creep on loading

$$s-s_0 = T_0(1-e^{-\mu t/\nu})/3\mu = \tfrac{2}{3}(1+\eta)(T_0/E)(1-e^{-\mu t/\nu}), \quad . \quad . \quad . \quad . \quad (9)$$

giving an exponential curve for the strain creep and a strain after an infinite time

$$s_\infty = T_0\{1+\gamma(1-2\eta)\}/E. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

On unloading we get a discontinuity which exactly cancels the discontinuity s_0 on loading. We thus get a *residual strain on unloading*

$$s'_0 = s_\omega - s_0 = \tfrac{2}{3}(1+\eta)(T_0/E)(1-e^{-\mu t_\omega/\nu}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

so that the initial strain on recovery should equal the total strain creep on loading.

At a time t' after unloading

$$s' = e^{-\mu(t_\omega+t')/\nu}\int_0^{t_\omega} e^{\mu t/\nu}T_0\,dt$$

$$= e^{-\mu t'/\nu}(s_\omega - s_0) = s'_0 e^{-\mu t'/\nu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Thus the strain creep on recovery $s'_0 - s' = s'_0\,(1-e^{-\mu t'/\nu})$, and, at corresponding times ($t' = t$)

$$\frac{\text{creep on recovery}}{\text{creep on loading}} = \frac{s'_0}{\tfrac{2}{3}(1+\eta)\,T_0/E} = 1-e^{-\mu t_\omega/\nu}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This ratio should therefore be invariable in the same experiment, and should approach unity the longer the load has been left on. Also the creep up to any time, as well as the initial strain, must be proportional to the load applied.

Coming now to the consideration of the stress-optical effect on this provisional theory, it was assumed that both the elastic and the viscous parts of the principal stress-difference each produced its own share of the relative retardation of the two oppositely polarized rays traversing the plate of celluloid, it being known that artificial double refraction is produced in a viscous fluid.*

We have, then, r being the relative retardation per unit thickness of the specimen

$$r = 2\mu A\,(s_1-s_2) + 2\nu B\,(\partial/\partial t)(s_1-s_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

A, B being the stress optical coefficients for the elastic and viscous parts of the stress respectively.

* CLERK MAXWELL, 'Roy. Soc. Proc.,' No. 148 (1873). See also 'Scientific Papers,' vol. 2, p. 379 (1890). A. KUNDT, 'Wiedemann's Annalen,' vol. 13, p. 110 (1881).

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But from (1) and (2), by subtraction,

$$(\mu+\nu\,\partial/\partial t)\,(s_1-s_2) = \tfrac{1}{2}\mathrm{T}, \qquad (15)$$

whence, as before,

$$s_1-s_2 = (1/2\nu)\,e^{-\mu t/\nu}\int_0^t e^{\mu t/\nu}\mathrm{T}\,dt,$$

and

$$r = \mathrm{BT}+(\mathrm{A}-\mathrm{B})\,(\mu/\nu)\,e^{-\mu t/\nu}\int_0^t e^{\mu t/\nu}\mathrm{T}\,dt. \qquad (16)$$

If we now examine equation (16) it is found to show generally characteristics precisely similar to those exhibited by the strain.

The initial retardation

$$\nu_0 = \mathrm{BT}_0.$$

The optical creep on loading

$$r-r_0 = (\mathrm{A}-\mathrm{B})\,\mathrm{T}_0\,(1-e^{-\mu t/\nu}). \qquad (17)$$

The residual retardation on unloading

$$r'_0 = r_\omega - r_0 = (\mathrm{A}-\mathrm{B})\,\mathrm{T}_0\,(1-e^{-\mu t_\omega/\nu}) \qquad (18)$$

The optical creep on recovery

$$r'_0-r' = (\mathrm{A}-\mathrm{B})\,\mathrm{T}_0\,(1-e^{-\mu t_\omega/\nu})\,(1-e^{-\mu t'/\nu}), \qquad (19)$$

and the ratio

$$\frac{\text{optical creep on recovery}}{\text{optical creep on loading}} = \left(\frac{r'_0-r'}{r-r_0}\right)_{t'=t} = 1-e^{-\mu t_\omega/\nu}, \qquad (20)$$

and so should be the same for the optical creep and for the strain creep.

In the above the retardations are in every case *per unit thickness* of the specimen.

(1) We note that, on this theory, the existence of an optical creep depends upon A and B being unequal, that is, on the stress-optical coefficients for the viscous and elastic parts of the stress being different.

(2) The existence of a sudden change of retardation on loading and unloading implies that B is not zero, that is the viscous part of the stress must be optically active.

(3) The creep, as well as the initial retardation, must be proportional to the load applied.

§ 7. *Comparison of Provisional Theory with Observation.*

The theory described above gives, on the whole, a fairly accurate *qualitative* description of the facts observed. When, however, the observations of optical creep came to be plotted to time and it was attempted to fit to the curves formulæ of the type

$$r = \alpha-\beta e^{-\kappa t}, \qquad (21)$$

it rapidly became apparent that no such formula could give even an approximate fit.

Fig. 5 shows a typical curve of optical creep on loading and recovery.

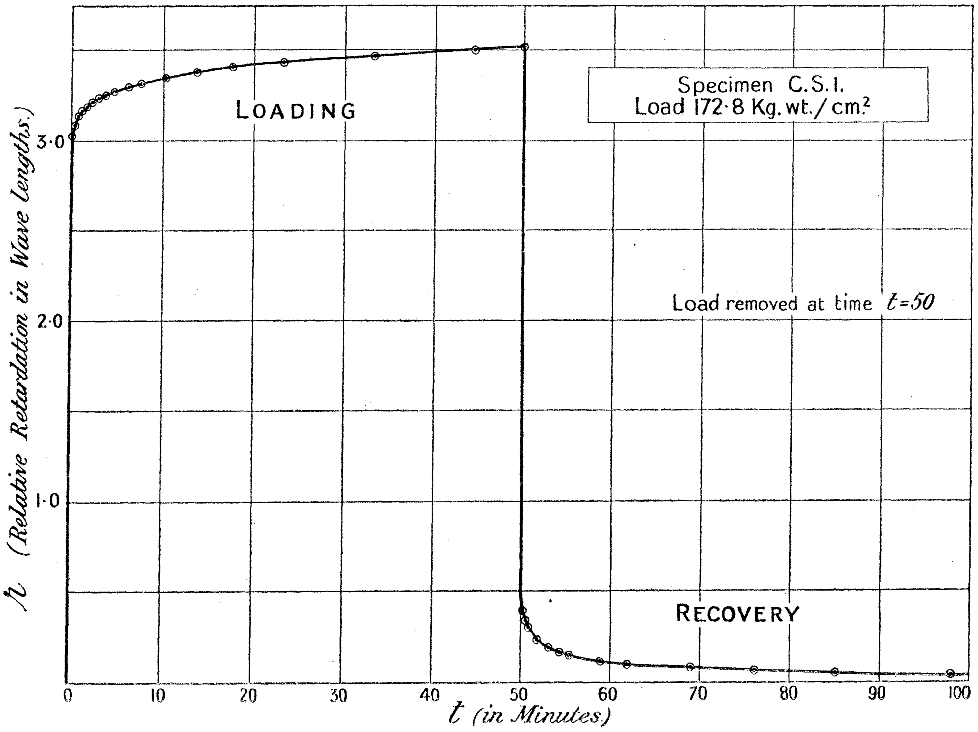

Fig. 5.

If these curves can be fitted by a formula of type (21), then

$$r_{t+\tau} = \alpha - \beta e^{-\kappa t - \kappa\tau},$$

$$r_t = \alpha - \beta e^{-\kappa t},$$

so that

$$r_{t+\tau} - r_t = \beta (1 - e^{-\kappa\tau}) e^{-\kappa t},$$

and

$$\log (r_{t+\tau} - r_t) = \log \beta - \kappa t + \log (1 - e^{-\kappa\tau}).$$

Thus, for τ constant, the values of $\log (r_{t+\tau} - r_t)$ plotted to the time should give a straight line of slope $(-\kappa)$.

Taking $\tau = 30$ minutes, the curve of $\log (r_{t+\tau} - r_t)$ is shown in fig. 6. This does not approximate to a straight line, even roughly. The value of k obtained from the slope of this curve varies from $0 \cdot 0625$ when $t = 2$ to $0 \cdot 0017$ when $t = 80$.

Fig. 7 shows values of $\log (r_{t+\tau} - r_t)$ plotted to the time for loading and recovery of another specimen. In both cases a straight line fit is impossible.

It is therefore quite obvious that the exponential formula is definitely inadmissible.

A very striking characteristic of all the observed creep and recovery curves was the extreme steepness of these curves *at the start*. This was so marked as to indicate that the initial value of dr/dt could be treated as practically infinite. The impression given by a careful study of the curve and by the fitting of a number of empirical formulæ was that the curve actually *touched* the initial ordinate at a definite point A, which corresponded to the initial value of the retardation. This also was incompatible with the exponential formula.

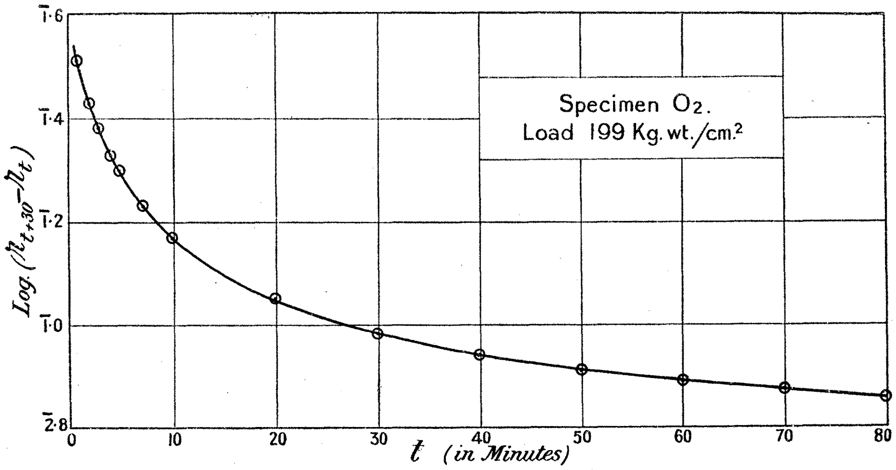

Fig. 6.

When observations for different loads were compared, it was found that these initial retardations were, within the errors of such determinations, *roughly proportional to the*

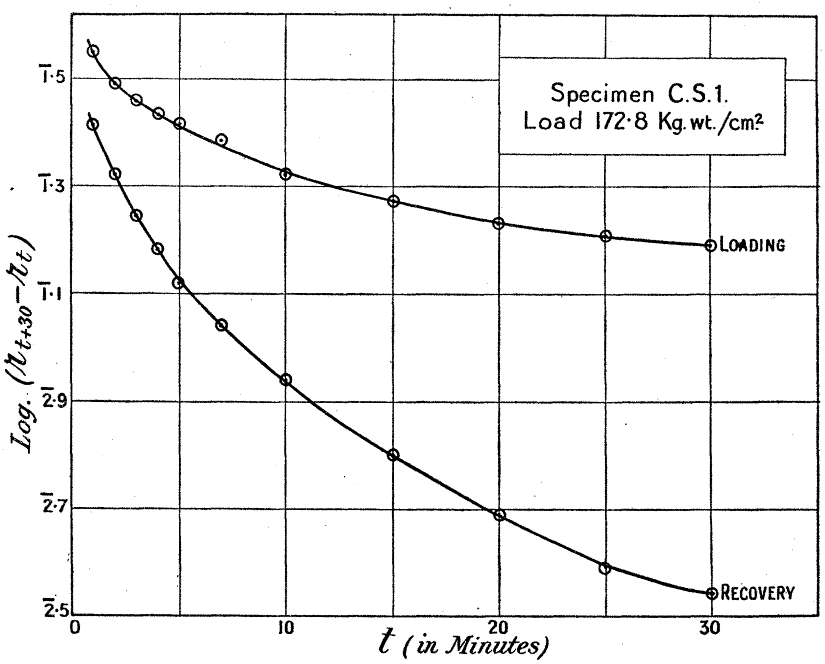

Fig. 7.

applied load. To this extent the results agreed with the provisional theory of § 6. But the latter is again contradicted by the values of the optical creep, which are not even roughly proportional to the applied load.

Table I. gives initial retardations and total optical creeps for various loads, the creeps being measured for the same time interval in all cases.

TABLE I.

Stress kg. wt./cm.^{2}	Initial retardation (wave-lengths).	$\dfrac{\text{Init. ret.}}{\text{Stress}}$.	Optical creep in 1 hour (wave-lengths).	$\dfrac{\text{Creep}}{\text{Stress}}$.
60	1·034	0·0172	0·101	0·00168
116·4	1·848	0·0159	0·289	0·00248
	1·834	0·0158	0·283	0·00243
	1·902	0·0163	0·194	0·00167
172·8	2·820	0·0163	0·650	0·00376
	2·770	0·0164	0·770	0·00446
	3·015	0·0174	0·660	0·00382
	2·940	0·0170	0·700	0·00405

On the other hand, the loading and recovery curves should, if the load has been on long enough, be superposable on turning round through 180°, since we should then have

$$r'_0 - r' = r - r_0,$$

or

$$r + r' = r_0 + r'_0 = \text{const.},$$

and this is independent of any error in the determination of r_0 and r'_0.

This condition is, on the whole, fairly well satisfied.

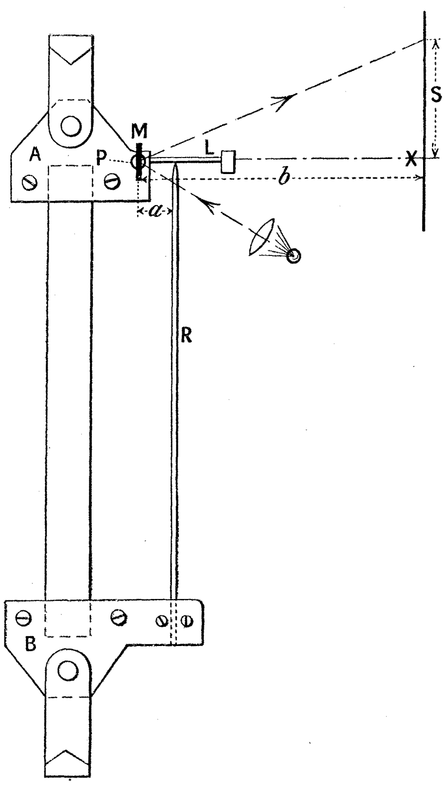

Fig. 8.

§ 8. *Method of Observation of the Strain in Celluloid.*

In view of the fact that the provisional theory of § 6 would not fit the observations we decided to observe directly the actual strain in the celluloid, and an extensometer was devised to enable us to measure this.

The ends of the celluloid specimen were securely fastened in two clamps A and B (fig. 8), the upper one being suspended from a fixed support, while the lower could be loaded by means of a lever and weights. Through the jaws of the upper clamp passed a horizontal spindle P, along the axis of which was mounted a small mirror M. The spindle was provided with a weighted arm L, which rested on the end of a light rod R, whose other end was rigidly secured to the bottom clamp B.

In order to reduce friction, the end of the rod was smoothly pointed, while a small glass plate was fixed to the underside of the radial arm L. The glass plate sliding on the steel point produced very little friction, and a gentle tap given to the supporting frame just before taking a reading ensured that no sticking had occurred.

A filament of an incandescent electric lamp was focussed after reflection from the mirror upon a vertical scale some distance away, on which the readings were taken.

In the experiments carried out the distance a of the rod from the mirror was 2·6 cm., while the distance b of the scale was 221 cm. This gave a magnification of 170 at the point X of the scale.

The corrections for vertical motion of the rod and for inclination of the ray to the scale were worked out and found to be in every case less than $\frac{1}{3}$ per cent., which was slightly less than the accuracy of the readings, so that it was considered unnecessary to apply these corrections.

§ 9. *Characteristics of the Strain-Time Variation.*

The strains were found capable of being measured by this method more accurately than the optical retardation, and the curves through the observations were very well determined, the average error of an observation of extension being 0·005 mm.

The general characteristics of the curves of extension to time were strikingly similar to those of the curves of optical retardation to time. One such curve of loading and recovery is given in fig. 9.

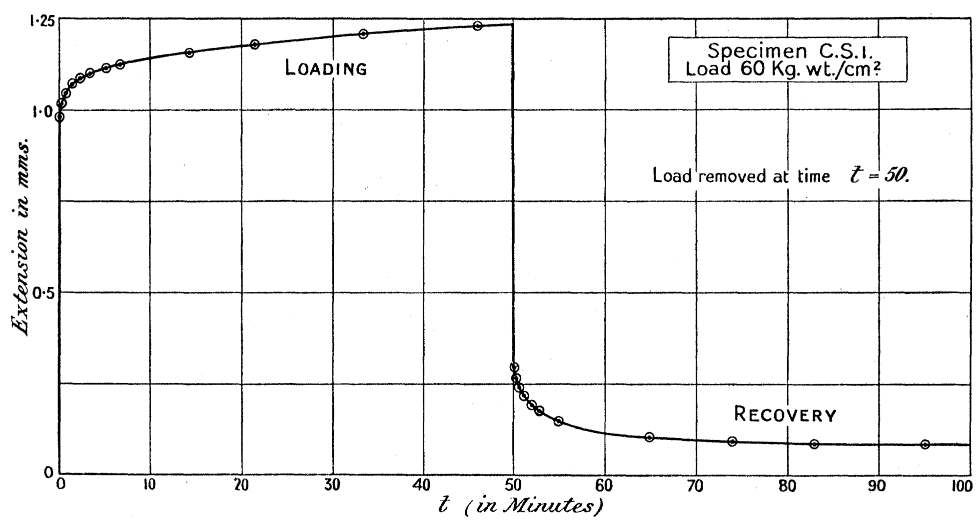

Fig. 9.

We found an initial extension and an initial strain recovery, as before, and a curve practically touching the ordinate at these points. The exponential formulæ of type (9) and (12) failed to fit, in the same way as for the optical creep.

The initial extension, like the initial optical retardation, was found roughly proportional to the applied load. The creep, however, seemed to show no very definite relation to the stress-optical creep, but in these earlier experiments the range of loads was different for the strain and retardation observations, being higher in the latter case.

Further, on repeating some of the experiments, it became evident that both the optical effect and the strain were seriously affected, as regards both the initial values and the creep, but more especially the latter, by previous treatment.

Tables II. and III. show the values of the retardation and strain for the same specimen, on being reloaded after a comparatively short rest. The results all agree in showing a slight diminution of the effect on repeated loading for the lesser load, but an increase on repetition for the higher load.

TABLE II.—Effect of Repeated Loading on Optical Retardation.

Load.	116·4 kg. wt./cm.²		172·8 kg. wt./cm.²		
Time in hours after loading.	Original loading.	After 24 hours.	Original loading.	After 4 hours.	After 24 hours.
0·01	1·984	1·968	3·120	3·130	3·150
0·02	1·998	1·980	3·160	3·170	3·200
0·0625	2·027	2·010	3·225	3·245	3·280
0·1	2·043	2·025	3·255	3·290	3·320
0·3	2·090	2·068	3·350	3·405	3·455
0·5	2·115	2·091	3·405	3·455	3·535
1·0	2·137	2·117	3·470	3·540	3·675
Total creep .	0·153	0·149	0·350	0·410	0·525

(Retardation in wave-lengths of sodium light.)

TABLE III.—Effect of Repeated Loading on Strain.

Extension in Scale Divisions.

Load.	39·5 kg. wt./cm.²		60 kg. wt./cm.²			80·5 kg. wt./cm.²	
Time in hours after loading.	Original loading.	After 24 hours.	Original loading.	After 4 hours.	After 24 hours.	Original loading.	After 24 hours.
0·01	11·2	9·65	16·2	16·1	16·6	20·6	21·5
0·02	11·4	9·83	16·5	16·4	17·0	21·0	21·9
0·0625	11·7	10·2	17·2	17·0	17·6	21·8	22·7
0·1	11·9	10·4	17·5	17·3	17·9	22·1	23·1
0·3	12·45	10·9	18·35	18·0	18·7	23·0	24·1
0·5	12·75	11·2	18·8	18·4	19·2	23·5	24·6
1·0	13·2	11·6	19·3	19·0	20·0	24·4	25·4
Total creep .	2·0	1·95	3·1	2·9	3·4	3·8	3·9

§ 10. *Simultaneous Observations of Strain and Retardation.*

On account of this "historical" effect the optical and strain observations, even when made on the same specimen, were not comparable unless they were taken simultaneously, and the apparatus was therefore modified so as to allow of this being done.

A link extensometer was devised which would measure changes of distance between two points on the specimen used for the stress-optical experiments. Fig. 10 shows a

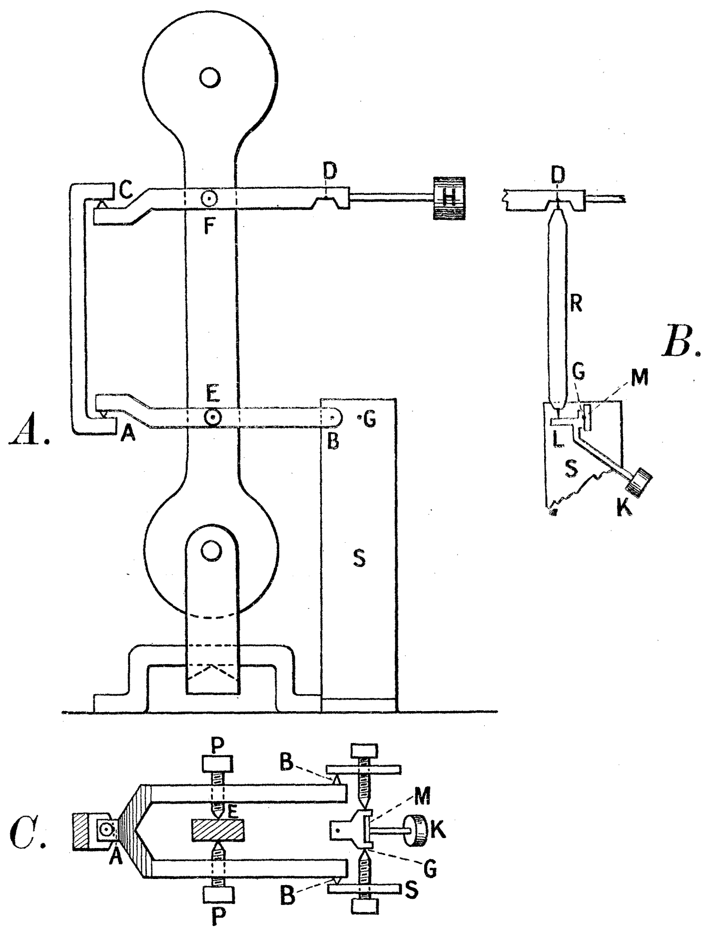

Fig. 10.

A. Elevation. B. Details of mirror and link. C. Plan.

diagrammatic sketch of the apparatus. Two equal links AB and CD were attached to the specimen at their midpoints E and F by means of two pairs of pointed screws P, P. The lower link was similarly pivoted about a fixed hinge B, while the ends A, C of the links were connected by a celluloid link AC of length equal to the distance EF. A counterpoise H held these three links in contact. Thus the absolute height of the point D would not be affected by up and down movement of the specimen as a whole.

Another fixed pivot G carried a mirror M (shown in a separate sketch for clearness) actuated by a lever L on which rested the lower end of a celluloid rod R (also of length equal to EF), whose upper end rested at the point D of the upper link. The rod R was fitted with needlepoints at its ends, and these points rested in star-holes, being held in position by the weight of the counterpoise K, attached to the mirror-lever.

By this arrangement changes in length due to variations in temperature were compensated, but any alteration in the length EF of the specimen due to stretching under load produced a rotation of the mirror, which was measured by means of a reflected ray focussed on a scale at a distance. Meanwhile the faces of the specimen were left clear for the observation of the optical effect.

A magnification of 1000 was obtained with this apparatus, which was considerably more sensitive than the one previously used. Also a specimen, once used, was not again put under strain until an interval of several weeks had elapsed, when it was found to have sensibly returned to its original condition.

Fig. 11 shows three curves obtained from one specimen under the same load: (*a*) on

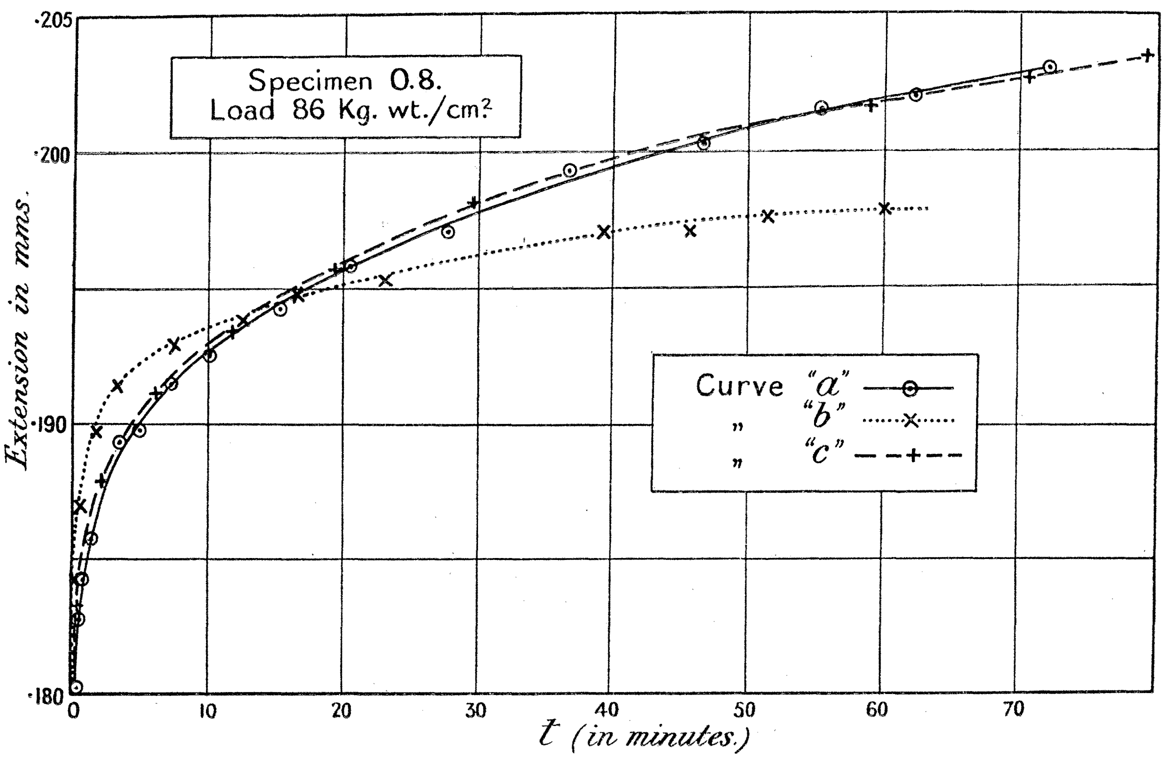

Fig. 11.

first loading, (*b*) on reloading after three hours' rest, (*c*) on reloading after ten days' rest. The curves (*c*) and (*a*) are seen to be practically identical.

§ 11. *Linear Relation between Retardation, Stress and Strain.*

When the optical retardation is plotted to simultaneous strain, it is found that in every case the results are closely fitted by a straight line, so that the retardation r, in Ångströms per millimetre is given by $r = \gamma + \beta S$, S being strain in 10^{-5} units.

Figs. 12A and 12B show a few typical cases, the straight line being in every case the best straight line fitted by least squares.

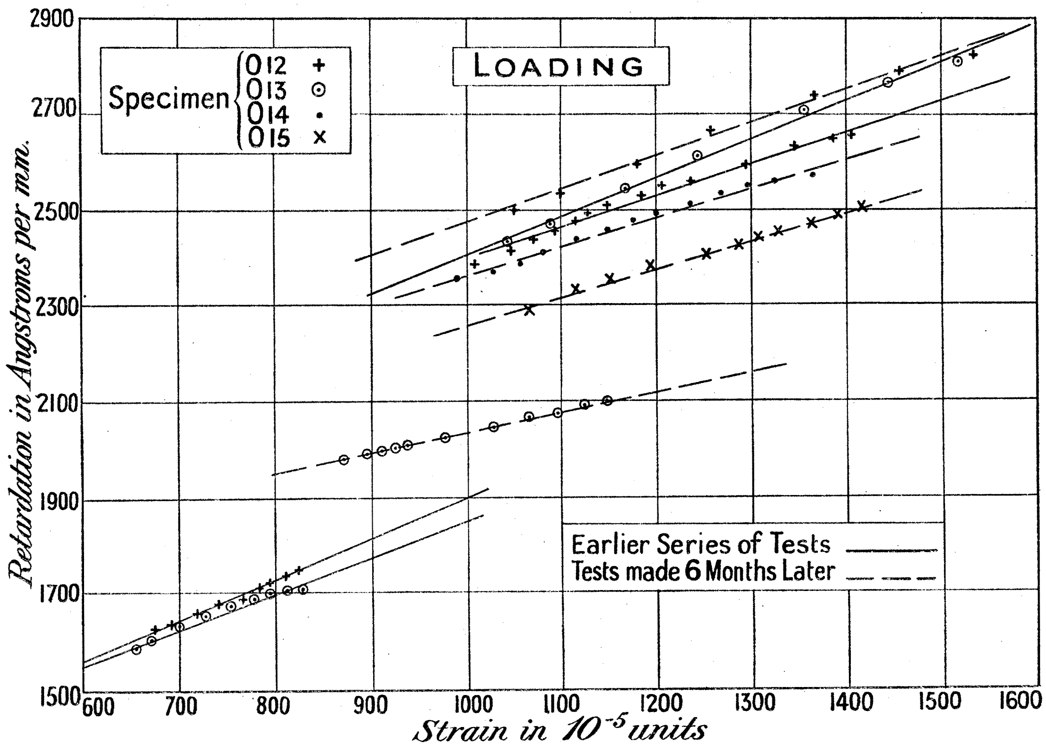

Fig. 12A.

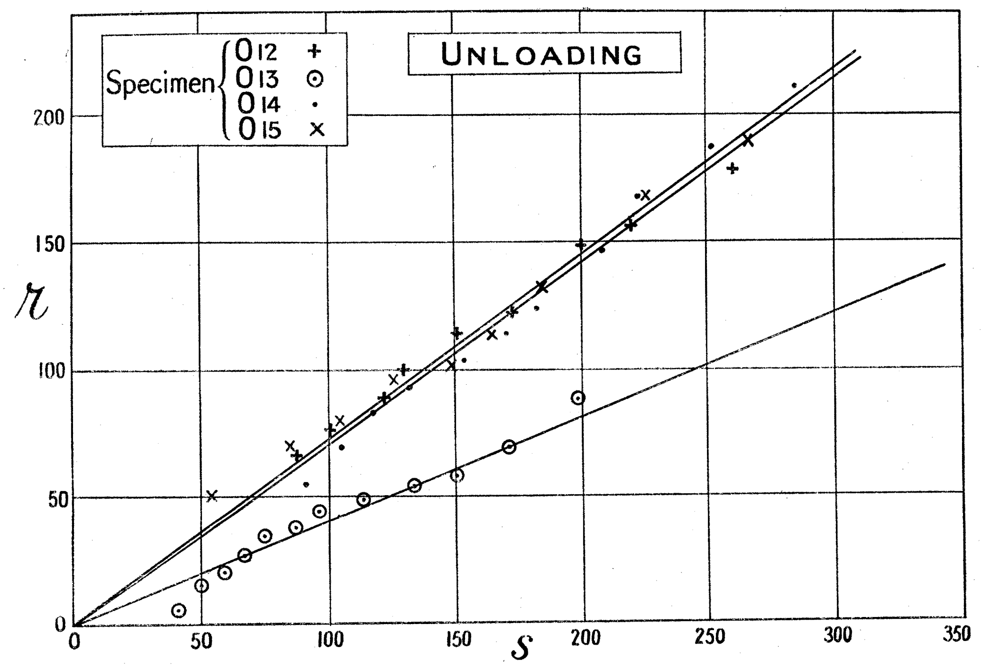

Fig. 12B.

The first curves considered were the loading curves for two specimens O_{12} and O_{13}, each tested under two stresses of approximately 200 and 135 bars.

The values of γ and β found were as follows :—

$$O_{12}\begin{cases} T = 135\cdot 4 \text{ bars}, & \gamma = 1059\cdot 10, \quad \beta = 0\cdot 8363 \\ T = 201\cdot 1 \quad ,, & \gamma = 1777\cdot 17, \quad \beta = 0\cdot 6942 \end{cases}$$

$$O_{13}\begin{cases} T = 134\cdot 4 \quad ,, & \gamma = 1112\cdot 42, \quad \beta = 0\cdot 7367 \\ T = 200\cdot 1 \quad ,, & \gamma = 1593\cdot 48, \quad \beta = 0\cdot 8111 \end{cases}$$

How close the fit is may be judged from Table IV. which gives the observed and calculated values of r for these four cases.

TABLE IV.—Retardations in Ångströms per Millimetre.

O_{12}.						O_{13}.					
T = 135·4.			T = 201·1.			T = 134·4.			T = 200·1.		
Obs.	Calc.	Res.	Obs.	Calc.	Res.	Obs.	Calc.	Res.	Obs.	Calc.	Res.
1627	1625	−2	2498	2506	+ 8	1585	1596	+11	2431	2439	+ 8
1639	1639	0	2534	2540	+ 6	1603	1606	+ 3	2471	2477	+ 6
1661	1661	0	2596	2596	0	1631	1625	− 6	2542	2540	− 2
1677	1678	+1	2665	2650	−15	1651	1644	− 7	2614	2602	−12
1697	1699	+2	2739	2726	−13	1675	1667	− 8	2706	2693	−13
1711	1714	+3	2787	2787	0	1688	1685	− 3	2766	2765	− 1
1725	1724	−1	2827	2841	+14	1700	1697	− 3	2810	2825	+15
1737	1736	−1	—	—	—	1706	1711	+ 5	—	—	—
1749	1746	−3	—	—	—	1714	1722	+ 8	—	—	—

The mean-square residual in the above is 7·4 Ångs./mm., corresponding to about 50 Ångströms total retardation. This is just about the order of the error of the observations, which was estimated to be just under 1 per cent. of a wave-length of sodium light.

A large number of cases have been examined, and in every case the mean-square residual was of the above order and frequently less.

In the above results we notice γ is roughly proportional to the stress. If we denote γ/T by α, α is measured in a unit of 1 Ångström per millimetre per bar. This unit has been called by one of us a brewster ; it is equal to 10^{-13} cm.^{2} per dyne and is very convenient for expressing stress-optical coefficients, since we then have :—

Retardation in Ångströms = thickness in millimetres × stress in bars × stress optical coefficient in brewsters.

The values of α in the four cases considered are :—

For O_{12},	$T = 135\cdot4$ bars,	$\alpha = 7\cdot8220$
	$T = 201\cdot1$,,	$\alpha = 8\cdot8373$
For O_{13},	$T = 134\cdot4$,,	$\alpha = 8\cdot2769$
	$T = 200\cdot1$,,	$\alpha = 7\cdot9634$

From the above we see that on the whole α and β appear to vary comparatively little from load to load, and even from specimen to specimen.

This suggests a relation of type

$$r = \alpha T + \beta S, \quad \ldots \quad (22)$$

which should hold for a wide range of values of T and S.

A formula of this type was fitted by least squares to the observations of O_{12} and O_{13}, α and β being now assumed to be *the same* for different experiments on the same specimen.

The following values were found :—

For O_{12}, $\alpha = 8\cdot3032$, $\beta = 0\cdot7687$

For O_{13}, $\alpha = 7\cdot8142$, $\beta = 0\cdot8296$

The values of r calculated from the formula were then compared with the observed. The comparison is given in Table V.

TABLE V.

T.	O_{12}. S.	r (obs.).	r (calc.).	Res.	T.	O_{13}. S.	r (obs.).	r (calc.).	Res.
135·4	677	1627	1645	+18	134·4	656	1585	1594	+ 9
	693	1639	1657	+18		670	1603	1606	+ 3
	720	1661	1678	+17		696	1631	1628	− 3
	740	1677	1693	+16		721	1651	1648	− 3
	765	1697	1712	+15		753	1675	1675	0
	783	1711	1726	+15		777	1688	1695	+ 7
	795	1725	1735	+10		794	1700	1709	+ 9
	810	1737	1747	+10		812	1706	1724	+18
	822	1749	1756	+ 7		828	1714	1737	+23
201·1	1050	2498	2477	−21	200·1	1042	2431	2428	− 3
	1099	2534	2515	−19		1089	2471	2467	− 4
	1179	2596	2576	−20		1167	2542	2532	−10
	1257	2665	2636	−29		1244	2614	2596	−18
	1367	2739	2721	−18		1355	2706	2688	−18
	1455	2787	2788	+ 1		1444	2766	2762	− 4
	1533	2827	2848	+21		1518	2810	2823	+13

The mean-square residuals are here very much larger than in the case where the sets of observations are fitted individually, being 17·1 Ångs./mm. for O_{12}, and 11·3 Ångs./mm. for O_{13}. Nevertheless they are not so large as to make the formula by any means an impossible or even a definitely bad fit.

If the formula (22) has a real physical significance, it should hold for retardations obtained *during unloading*, in which case we ought to find

$$r = \beta S,$$

i.e. the observations of retardation to strain should be fitted by a straight line through the origin.

Unfortunately, no observations of the recovery curves had been taken at the time, and pressure of other work had necessitated the dismantling of the apparatus. Hence several months elapsed before it was possible to repeat the experiments on the two specimens, so as to note the values on unloading.

When this was done, a very curious result appeared in the case of O_{13}. The α and β for the *loading* curve for a stress $T = 167{\cdot}5$ bars *intermediate between the two previously used*, came out widely different, thus $\alpha = 9{\cdot}5913$, $\beta = 0{\cdot}4272$.

On *unloading*, a straight line was found to give a very close fit. The *best* straight line, however, did not go exactly through the origin, giving

$$r = -7{\cdot}946 + 0{\cdot}4756S.$$

We have worked out the probable errors of the constants γ and β given by the least square solution, in this case, and they come to about 5 Ångströms per millimetre, if the probable error of a retardation reading is about 60 Ångströms, which is probably an underestimate for very small residual retardations. Under these conditions the constant term in the above can hardly be regarded as significant. Neglecting it, and fitting a formula $r = \beta S$ to the observations, we find $\beta = 0{\cdot}4083$. The fit is nearly as good, giving a mean square residual of only 5·3 Ångströms per millimetre.

Hence the β on unloading is very sensibly equal to the β on loading, and this would give confirmation of the general formula $r = \alpha T + \beta S$, were it not for the considerable change in the constants α, β, from the previous observations.

This suggests that the lapse of time has, in some way which we cannot definitely explain, affected the optical coefficients of the specimen. Possibly a drying or hardening process has affected its constitution.

In the case of O_{12} a heavier load was used giving $T = 225{\cdot}2$ bars. Here the constants on loading came out to be $\alpha = 7{\cdot}6432$, $\beta = 0{\cdot}6749$. This is in much better agreement with the two previous results for the same specimen. Both the constants have been slightly reduced. The *unloading* curve gave a closely fitting straight line

$$r = 9{\cdot}0946 + (0{\cdot}6660)\,S,$$

or, if we fit so as to make the straight line pass through the origin (treating the 9·09 Ångs./mm. as negligible),

$$r = (0 \cdot 7205)\, \mathrm{S},$$

which again gives a value of β in fair agreement with the loading curve, and also with the previously found values of β for this specimen.

It would appear, therefore, that the law $r = \alpha\mathrm{T} + \beta\mathrm{S}$ is a fairly good fit, the behaviour of O_{13} being anomalous.* This accounts for the remarkable similarity which undoubtedly exists between the curves of extension to time and those of optical retardation to time, under constant load. In the earlier experiments in which these two were observed separately, this similarity was often partly masked by the influence of previous treatment on the specimens, and by the fact that the same specimen could not be employed for both sets of observations without reshaping.

§ 12. *Comparison with the Results of* Coker *and* Chakko.

Meanwhile it is of considerable interest to examine the results of Coker and Chakko (*loc. cit.*). Table I., on p. 149 of their memoir, gives a set of corresponding values of strain, stress, and optical retardation; the latter, however, being expressed as an "Equivalent Stress" deduced from the simple stress-optical law, so that their units are not comparable with those used in this paper. Using, however, Coker and Chakko's units, we find, in their notation

$$f_0 = 94 \cdot 5 + 0 \cdot 426296 f + 154642 e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Now f_0 is a multiple of our retardation, r; f is the stress in lbs. weight per square inch, so that $f = 14 \cdot 499\mathrm{T}$ where T is in bars, and e is the strain in absolute number $= 10^{-5}\mathrm{S}$ if we use the same unit of strain as in § 11.

This leads to

$$\mathrm{K}r = 94 \cdot 5 + 6 \cdot 1808\mathrm{T} + 1 \cdot 5464\mathrm{S}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where K is some constant.

The first term, 94·5, corresponds to an initial stress and optical effect in the material before loading. This initial stress, independently determined, was found by Coker and Chakko to be 73 lbs. weight per square inch, and the agreement is quite within the probable error of the observations.

* A possible explanation of the marked alteration in the coefficients of O_{13} may be found in the fact that in the last of the earlier tests this specimen was subjected to a load of about 236 bars, which was considerably greater than the heaviest load applied to any other specimen, and may have produced serious *permanent strain*. This could not have been detected, as the extensometer was dismantled between the tests, and there was no means of recovering the zero. Thus, although the recovery of the stress-optical effect was complete (and this could be ascertained at any time by means of the compensator), there may have remained considerable *strain* which would have affected the new strain readings for the specimen.

Table VI. gives a comparison of a few of the later values of COKER and CHAKKO's Table I., when stress, strain and optical retardation are no longer proportional, with our formula (23).

TABLE VI.

Strain. $e = S \times 10^{-5}$.	Stress. f (lbs. per square inch).	Optical retardation measured as equivalent stress. f_0 (lbs. per square inch).	
		Observed.	Calculated.
0·0037	1115	1173	1142
0·0077	2230	2200	2236
0·0117	3185	3230	3262
0·0135	3505	3560	3676
0·0153	3825	4140	4091
0·0172	3980	4500	4451
0·0187	4140	4770	4751
0·0208	4300	5100	5144

On an average the discordance between observation and formula is of the order only of about 1 per cent., and this is probably the order of accuracy of the optical observations. Further, it has to be remembered that corresponding readings in COKER and CHAKKO's table are probably not exactly simultaneous, and, as they kept adding to their load, both their measured strain and their optical effect must have been affected by creep, as described in the present paper. We may take it, therefore, that their observations are in substantial agreement with a formula of type (22).

The absolute values of α and β cannot be expressed in our units, as the constant K in (24) cannot be calculated from the data.

The ratio $\alpha : \beta$ is found to be $3 \cdot 997$ for COKER and CHAKKO's specimen. For most of our observations the same ratio is between 9 and 12.

There is evidence, however, that this ratio varies considerably from specimen to specimen. Thus for one specimen, O_8 we found $\alpha = 5 \cdot 46$, $\beta = 1 \cdot 58$, so that $\alpha : \beta$ is approximately $3 \cdot 456$, which is fairly close to the value for the sample tested by COKER and CHAKKO.

§ 13. *Theoretical Explanation of the Law* $r = \alpha T + \beta S$.

At first sight it might appear that the law established in the last two sections definitely negatives the hypothesis which we originally set out to test, namely, that *stress* and not *strain* is the determining factor in producing artificial double refraction. On the other hand, it appears equally opposed to the hypothesis which makes strain alone the essential factor. A little consideration, however, shows that, although the second hypothesis certainly cannot be reconciled with the formula, the same is not necessarily true of the first hypothesis.

The linear law

$$r = \alpha T + \beta S$$

admits of a simple physical interpretation if we suppose the material made up of two components, one perfectly elastic, and the other plastic.

Let b_1 = total thickness of the elastic part, measured in the direction of travel of the light.

b_2 = total thickness of the plastic part.

C_1 = stress-optical coefficient of the elastic part.

C_2 = ,, ,, ,, plastic part.

T_1 = tension of the elastic part.

T_2 = ,, ,, plastic part.

σ_1 = area of cross-section of the elastic part.

σ_2 = ,, ,, ,, plastic part.

Then if we suppose that for each kind of material the relative optical retardation is directly proportional to the *stress*, we have, ρ being the retardation in Ångströms,

$$\rho = C_1 T_1 b_1 + C_2 T_2 b_2. \quad \ldots \quad (25)$$

But also, if $\sigma = \sigma_1 + \sigma_2$ = total area of cross-section, and T = mean total tension (which is the observed tension),

$$\sigma T = \sigma_1 T_1 + \sigma_2 T_2 \quad \ldots \quad (26)$$

Further, S being the observed stretch and E_1 Young's modulus for the elastic part,

$$T_1 = E_1 S. \quad \ldots \quad (27)$$

We have then, eliminating T_1 and T_2 between the equations (25), (26) and (27), and calling b the total thickness $b_1 + b_2$,

$$\begin{aligned} r &= \text{retardation in Ångströms per millimetre.} \\ &= \rho/b = E_1 (C_1 b_1/b - C_2 b_2 \sigma_1/b\sigma_2)\, S + (C_2 b_2 \sigma/b\sigma_2)\, T, \quad \ldots \quad (28) \\ &= \alpha T + \beta S, \end{aligned}$$

where

$$\alpha = C_2 b_2 \sigma/b\sigma_2, \qquad \beta = E_1 (C_1 b_1/b - C_2 b_2 \sigma_1/b\sigma_2).$$

It appears then that such a heterogeneous composition of celluloid will account for the facts observed. This heterogeneity may be conceived as distributed uniformly throughout the material; the latter would then consist of a fine-grained elastic skeleton, of which the interstices are filled by a plastic or viscous magma.

Or we may take the view that the elastic and plastic parts are definitely separated from one another, *e.g.* the specimen may be looked upon as consisting of a plastic core enclosed in an elastic skin. That a skin where the optical effects are quite different from what they are in the core exists in celluloid is asserted by Coker and Chakko (*loc. cit.*), and any casual examination confirms this. Any exact direct determination of the stress-optical properties of this skin is, however, most difficult, the variations of initial internal stress in this skin being sufficient to blur the bands and render observation impossible near the edges.

In order to test definitely the hypothesis that the effect in question is due to the skin, two specimens, O_{14} and O_{15}, were cut from the same region of a plate of xylonite, so as to ensure, as far as possible, that their composition and general properties were identical.

The specimen O_{14} was then tested under a stress of 225·2 bars. It gave the usual straight line graph for retardation to strain, viz. :—

$$r = 7\cdot7673\mathrm{T}+0\cdot6122\mathrm{S} \text{ on loading,}$$

and

$$r = 0\cdot7140\mathrm{S} \text{ on unloading,}$$

giving values α, β fairly close to those observed for O_{12}.

The second specimen O_{15} had a thickness of 1 mm. removed from each of its faces, and was immediately placed in the apparatus and tested under a stress of 224·2 bars. Now if the "skin" in celluloid is scraped off, several hours, or even days, normally elapse before the surface again hardens. Hence, if the effect in question is due to the skin, a considerable variation was to be expected.

The test again gave straight line graphs, as follows :—

$$r = 7\cdot4316\mathrm{T}+0\cdot5911\mathrm{S} \text{ on loading,}$$

and

$$r = 0\cdot7234\mathrm{S} \text{ on unloading.}$$

The divergences in the constants from O_{14} are less than many which we have found to occur between different sets of observations on the same specimen.

It would seem therefore that we cannot account for the facts observed on this theory if we rely on the skin and core to produce the effect observed.

We are thus driven to the conclusion that if the explanation above given is correct, xylonite is an intimate mixture of two materials with different elastic and plastic properties and different stress-optical coefficients. This does not appear, in itself, inherently improbable ; and it is further possible that the two materials are allotropic modifications of the same substance, and that their proportions are altered by the action of stress and other causes. This might account for the observed divergences in the values of α and β under different conditions.

§ 14. *Analysis of Strain-Time Curves.*

We come now to the analysis of the individual strain-time and retardation-time curves under constant load. This has proved one of the most troublesome parts of the whole investigation. As already explained in § 7, an exponential formula of type (21), which was indicated by a mixed elastic-viscous theory, entirely failed to fit the facts. A variety of possible laws connecting S and t were tried, with greater or less success.

The fact that initially S was finite, but $\dot{S}$ infinite suggested

$$S = S_0 + at^n,$$

n being an exponent less than unity.

It was found that $n = \frac{1}{3}$ gave a fair fit for the whole range of the curve, a result which confirms the observations of Dr. E. N. DA C. ANDRADE on lead wires (' Roy. Soc. Proc.,'

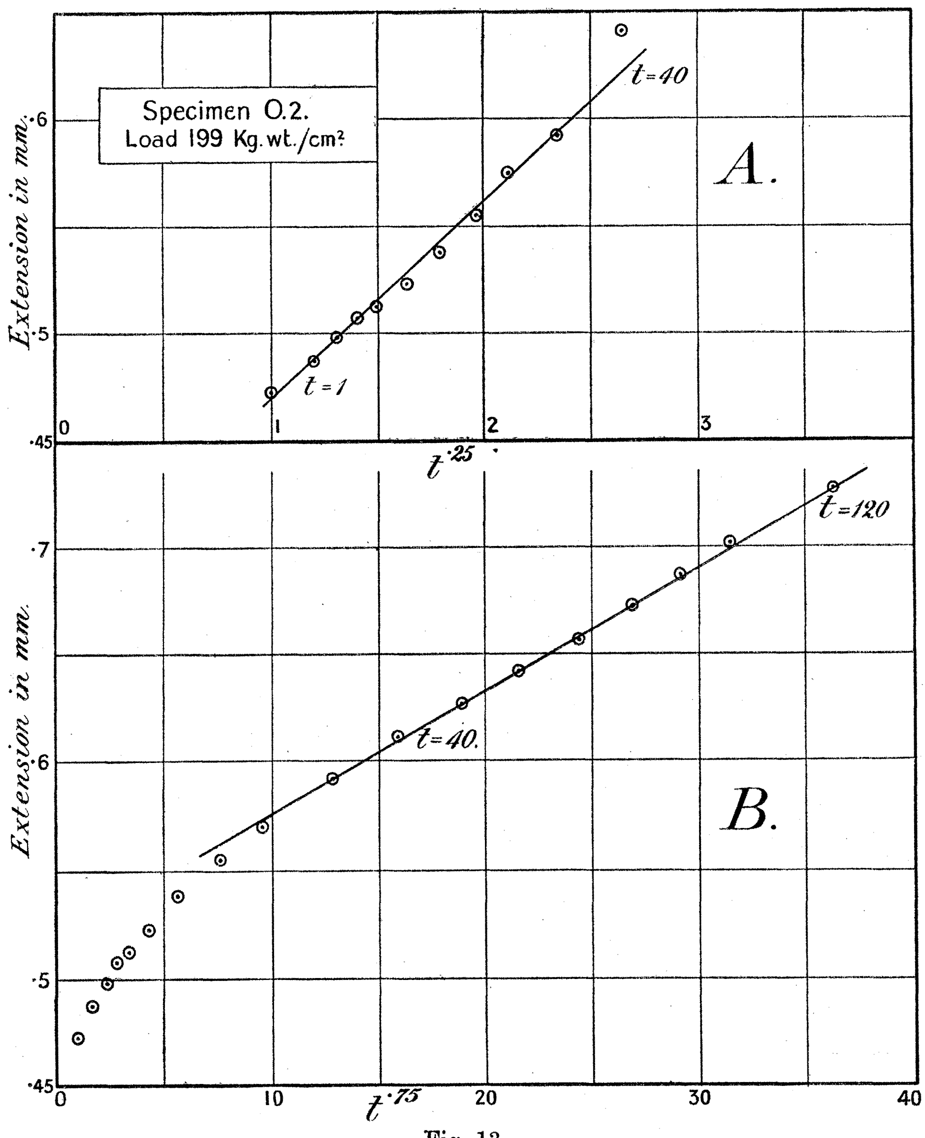

Fig. 13.

A, vol. 84, pp. 1–12, 1910), but for the earlier part of the curve a different value of n, approximating to $n = 0 \cdot 25$ or even less, gave a very good fit, while the latter part was fitted well by a value between $0 \cdot 5$ and $0 \cdot 75$. Fig. 13 illustrates this. It shows the strains plotted to $t^{1/4}$ and $t^{3/4}$ respectively. The observations for $t < 40$ lie fairly well on a straight line in the first case; those for $t > 40$ on another straight line in the second case, which, however, fails to fit the observations for $t < 40$.

This divergence between the best values of n for the earlier and later parts of the curve, which repeated examinations of many specimens showed to be quite systematic, led us finally to reject this formula.

We then tried a succession of formulæ, introducing combinations of the logarithm, exponential and power. A list is given below.

$$S = a + bt \log t, \qquad \text{(i.)}$$

$$S = S_0 + t^n (a + b \log t), \qquad \text{(ii.)}$$

$$S = S_0 + (a - S_0)(1 - e^{-\kappa t}) + bt^n, \qquad \text{(iii.)}$$

$$S = (t + t_0)^{1/3} (a - be^{-\kappa t}), \qquad \text{(iv.)}$$

$$bt = S + m + a \log (S + n), \qquad \text{(v.)}$$

$$bt = S + m + a/(S + n). \qquad \text{(vi.)}$$

All these had ultimately to be rejected, for various reasons: (i.), (ii.) and (iii.) gave unsatisfactory fits; (iv.) fitted very well for large values of t, but showed systematic divergences for the smaller values; (v.) and (vi.) gave a very good all-over fit, but exhibited a curious indeterminacy in their constants, widely divergent sets of constants giving an almost identical curve, so that these constants were obviously unsuitable elements to define the physical characteristics of the flow. Moreover, the fact that formulæ (ii.) to (vi.) all involve as many as *four* constants, somewhat reduces the significance of the fit.

Finally, it occurred to us to try to modify the form

$$S = S_0 + at^n$$

by adding another term of simple form.

We tried accordingly

$$S = S_0 + at^n + bt.$$

Such a formula allowed of being easily fitted by a least-square method, and it gave an extraordinarily good fit for *either* of the two values $n = \frac{1}{2}$ and $n = \frac{1}{3}$. The following set of values is typical :—

TABLE VII.

Specimen O_{13}. T = 236·4 bars.

Time in minutes.	S. Obs.	$n = \frac{1}{2}$.		$n = \frac{1}{3}$.	
		Calc.	Diff.	Calc.	Diff.
1	1424	1428	+4	1423	−1
2	1487	1489	+2	1489	+2
3	1538	1534	−4	1536	−2
4	1576	1573	−3	1574	−2
5	1610	1607	−3	1607	−3
10	1732	1732	0	1731	−1
20	1897	1903	+6	1900	+3
30	2029	2026	−3	2029	0

The mean-square residual for $n = \frac{1}{2}$ is 3·52 and for $n = \frac{1}{3}$ is 2·0. In either case the fit is well within the error of observation.

We decided finally to adopt the formula

$$S = S_0 + at^{1/3} + bt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

not only because it gave a slightly better fit, but because it fitted in better with Dr. ANDRADE'S observations on lead (*loc. cit.*). We tried to fit the formula

$$S = S_0 + at^{1/2} + bt$$

to Dr. ANDRADE'S observations, and the fit failed. The formula with $n = \frac{1}{3}$, however, fits Dr. ANDRADE'S observations very closely.

The results of a least-square fit to the observations of O_{12}, O_{13}, O_{14}, O_{15}, are given below in Table VIII.

The probable divergence between the formula and the observations is here about 0·97, or about 0·1 per cent. of the observed values of S. This is well within the probable error of the observations themselves. In fact the fit is so close that the divergence cannot be shown on a diagram drawn to a scale suitable for reproduction.

TABLE VIII.—Fitting of Formula $S = S_0 + at^{1/3} + bt$ to Observations of Strain-time Creep.

Specimen.	O_{12}.						O_{13}.										O_{14}.		O_{15}.	
T	135·4		201·1		225·2*		67·6		134·4		167·5*		200·1		236·4		225·2		224·2†	
S_0	622·9		898·1		936·8		305·3		599·5		780·4		895·1		1180·9		914·1		1004·4	
a	60·269		155·080		110·042		18·723		55·878		91·110		150·272		239·020		112·334		107·600	
b	−0·7083		+2·5578		+0·6496		−0·3558		+0·1199		+0·0782		+2·6976		+3·5016		−0·4804		+0·1257	
Time in Minutes.	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.
1	681	682	1050	1056	1046	1047	323	324	655	656	870	872	1045	1048	1424	1423	1025	1027	1109	1112
2	697	697	1102	1099	1077	1077	329	328	670	670	896	895	1091	1090	1487	1489	1058	1057	1141	1140
3	709	708	1132	1129	1097	1097	331	331	680	681	912	912	1122	1120	1538	1536	1078	1078	1161	1159
4	715	716	1157	1155	1115	1114	333	334	690	689	926	925	1147	1145	1576	1574	1095	1094	1176	1175
5	725	722	1176	1176	1130	1128	337	336	696	696	938	937	1165	1166	1610	1607	1109	1109	1188	1188
10	746	746	1257	1258	1182	1180	341	342	721	721	977	978	1244	1246	1732	1731	1160	1161	1234	1235
20	770	772	1367	1370	1246	1248	349	349	753	754	1028	1029	1355	1357	1897	1900	1228	1229	1292	1294
30	788	789	1455	1457	1297	1298	353	353	776	777	1065	1066	1444	1443	2029	2029	1277	1278	1335	1335
40	802	801	1533	1531	1341	1339	355	355	796	795	1096	1095	1517	1517	—	—	1318	1318	1368	1367
Mean Square Residuals.	1·4		2·9		1·4		0·7		0·8		1·1		1·7		1·9		1·0		1·5	

* After several months' interval.

† Specimen with "skin" removed.

§ 15. *The Significance of the Constants in the Formula for the Strain.*

It is worth while to inquire into the physical significance of the formula (29).

We find in the first place that the initial strain, S_0, is in every case very nearly proportional to the applied load, at any rate for those observations taken at the same time.

TABLE IX.—Values of S_0/T.

Specimen.	T.	S_0/T.	Specimen.	T.	S_0/T.
O_{12}	135·4	4·601	O_{13}	67·6	4·516
O_{12}	201·1	4·466	O_{13}	134·4	4·461
O_{12}	225·2	4·164*	O_{13}	167·5	4·659*
O_{14}	225·2	4·059	O_{13}	200·1	4·473
O_{15}	224·2	4·480†	O_{13}	236·4	4·995

* After several months' interval.

† Specimen with "skin" removed.

On the whole, lapse of time does not appear to have very seriously affected this constant in the case of O_{13}. It has appreciably reduced it in the case of O_{12}. In the case of O_{15}, the process of removing the skin has somewhat increased S_0, as indeed one might expect if the skin is harder than the remainder, so that its removal decreases the effective YOUNG'S modulus. Even here the increase in S_0 is only about 10 per cent.

It will be noticed that the value of S_0/T for O_{13} under load 236·4 is considerably above the value for any other test, and this rather confirms the theory previously given that during this test the specimen was seriously overstrained.

It appears then that there exists a fairly definite *initial* YOUNG'S modulus, E_0, for this material, and $1/E_0 = S_0/T =$ about 4·4. This YOUNG'S modulus appears to vary somewhat with the load, also with time and with mechanical treatment of the specimen, but the changes do not seem to exceed 10 per cent.

The next important constant is a. This measures the bulk of the strain creep, and seems to be a well-determined constant. Roughly speaking, it seems to be comparable with the square of the load. Table X. gives the values of a/T^2.

TABLE X.

Specimen.	T.	a/T^2.	Specimen.	T.	a/T^2.
O_{12}	135·4	0·0033	O_{13}	67·6	0·0041
O_{12}	201·1	0·0038	O_{13}	134·4	0·0031
O_{12}	225·2	0·0022*	O_{13}	167·5	0·0032*
O_{14}	225·2	0·0022	O_{13}	200·1	0·0038
O_{15}	224·2	0·0021†	O_{13}	236·4	0·0043

*, †, as in Tables VIII. and IX.

Here the effect of the lapse of time does not seem to have seriously affected O_{13}, but in the case of the other three specimens, a has been reduced to something like half the value previously found. It has to be remembered that O_{14} and O_{15} had not been previously tested, but the plates from which all the specimens were cut had been originally supplied at the same time, so it is not unreasonable to suppose that any progressive change is likely to have affected them all more or less equally. The behaviour of O_{13}, on this view, would be exceptional, and this again confirms the anomalous behaviour of this specimen in connection with the stress-optical law, previously noted.

It appears then that the strain creep generally increases with the load more nearly as the square than as the first power, so that its proportional effect is greater for the higher loads.

Finally, as regards b. This constant is in every case small, and the probable error in its determination is considerable. No great weight can be attached to its variations, and it seems to be considerably affected by time and treatment. One conclusion from Table VIII. is that negative values of b seem (with the single exception of O_{14}) to occur only for the smallest load.

Now a positive b implies (if the formula is assumed to hold good indefinitely) that the creep will continue without limit until the specimen breaks, a result which is known to be true for most plastic materials if the load applied be sufficiently great. b is then the ultimate rate of creep. On the other hand, if b be negative, then

$$\dot{S} = \tfrac{1}{3}at^{-2/3}+b,$$

and vanishes when $t = \left(-\dfrac{a}{3b}\right)^{3/2}$ so that the strain increases up to a finite limit

$$S_0 + \frac{2}{3\sqrt{3}}\frac{a^{3/2}}{(-b)^{1/2}},$$

and then stops, so that a negative value of b implies a finite creep and time of creep. The quantity $\dfrac{2}{3\sqrt{3}}\cdot a^{3/2}/(-b)^{1/2}$ may be called the total creep and denoted by c.

In every case the lapse of time has seriously diminished the positive b, and in the case of O_{15}, the removal of the skin has apparently so weakened the specimen as to reverse the sign of b, which is negative for O_{14}, even for the high load of 225 bars.

§ 16. *The Strain Recovery.*

The equation of motion of the specimen may be found by eliminating t between the expressions for S and $\dot{S}$. We then get an equation connecting the rate of creep with the configuration of the material, and from this it is possible to derive, by making various assumptions, equations for the recovery.

(i.) We will first assume a finite limit to the creep, so that S has an ultimate value

$S_\infty = S_0 + c$. Then if we suppose the rate of flow to depend in each case, on the divergence of the actual from the equilibrium configuration,

$$\dot{S} = f(S_\infty - S) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where f is a positive function of its argument and

$$f(0) = 0, \qquad f(c) = \infty.$$

Using accents to distinguish quantities observed during recovery

$$\text{Rate of flow} = -\dot{S}'.$$

Divergence of actual from equilibrium strain $= S'$, if we assume that the recovery is *complete*. This is certainly the case for moderate loads, to which the present case applies.

We have then

$$-\dot{S}' = f(S'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Also all the curves show an infinite slope at the start, for recovery as well as for loading. Hence if S'_0 is the initial strain on recovery

$$\infty = f(S'_0),$$

so that, generally,

$$S'_0 = c.$$

If we write in (30)

$$S = S_\infty - u,$$

then

$$t = -\int_c^u \frac{du}{f(u)} = \int_u^c \frac{du}{f(u)},$$

and

$$t' = -\int_c^{S'} \frac{dS'}{f(S')} = \int_{S'}^c \frac{dS'}{f(S')}.$$

Whence

(*a*) The initial strain on recovery should be constant and equal to the total creep, whatever the stage at which unloading takes place.

(*b*) The loading and recovery curves are superposable, on turning through 180°.

(ii.) We will next assume that the creep is unlimited but that there is a limiting rate of creep b. In this case the equation of motion on loading can be suitably written in the form

$$\dot{S} = b + f(S - S_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where $f(\infty) = 0$, $f(0) = \infty$, and f is a positive function of its argument.

On this assumption the rate of flow is determined by *the departure from the initial configuration*, *i.e.* by the strain creep.

If the same equation is to hold for recovery

$$-\dot{S}' = b' + f(S'_0 - S') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Here we have to remember that $\dot{S}' = 0$ for some value of $S' \geq 0$, since the specimen can never recover *beyond* the original unstrained position.

Therefore we must have

$$b' + f(c') = 0$$

where $S'_0 > c' > 0$.

This implies that b' is *negative* and therefore *different from* b. Hence the relation between rate of flow and strain creep cannot be the same for loading and recovery.

If we write $S - S_0 = u$, $S'_0 - S' = u'$, then we have

$$t = \int_0^u \frac{du}{b + f(u)}, \qquad t' = \int_0^{u'} \frac{du'}{b' + f(u')}.$$

Hence, even if the function f is the same in both cases, the relation between t and u is not the same as that between t' and u', so that the recovery curve will be different from the loading curve. The difference, however, will clearly exist at whatever stage we remove the load, for b' being never positive, cannot under any circumstances approximate effectively to b.

The observed recovery curves do not fit in with either hypothesis (i.) or (ii.). Fig. 14

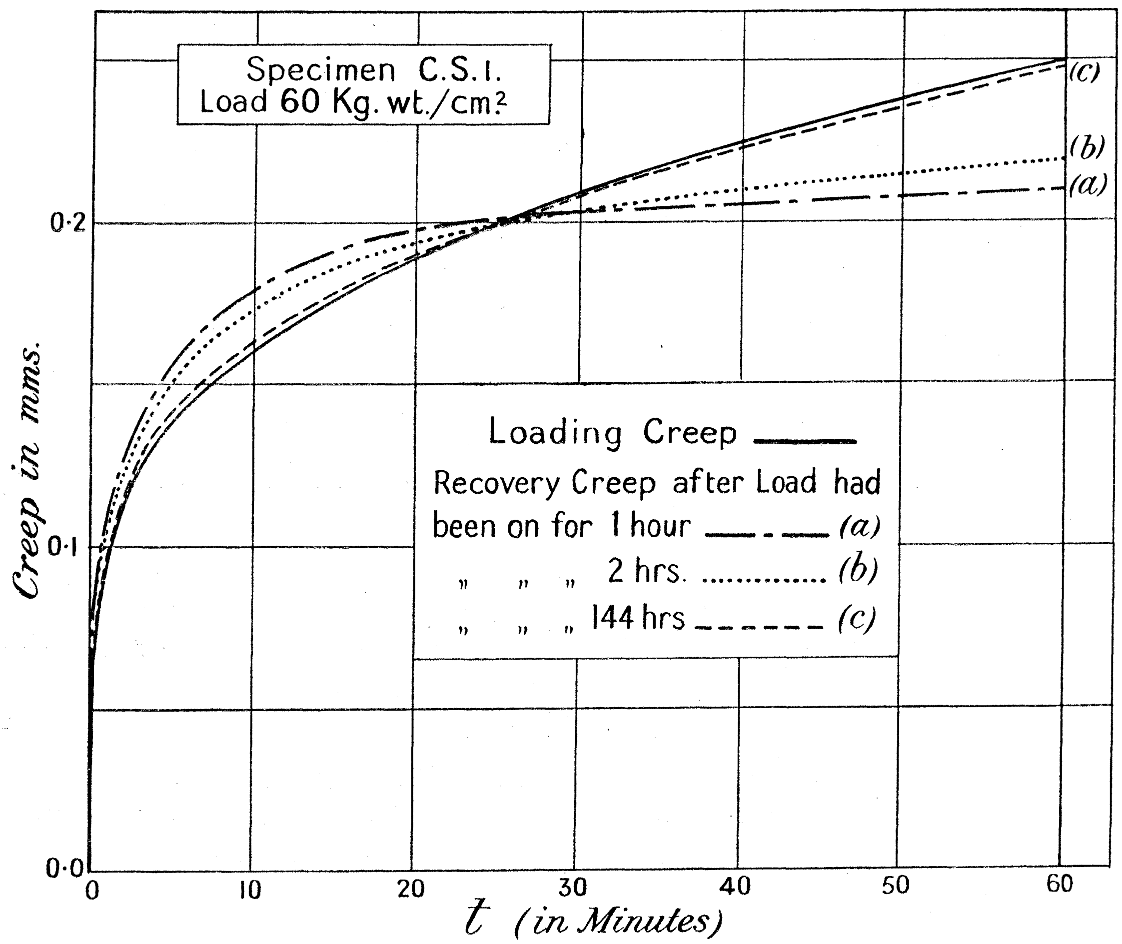

Fig. 14.

shows three typical recovery curves. They have been reversed and placed so that their starting points agree with that of the loading curve for the same specimen. The curve (*a*) was obtained when unloading after one hour; (*b*) after two hours; (*c*) after 144 hours. It is at once clear from the figure that (*c*) agrees very closely with the loading curve, but (*a*) and (*b*) can by no means be harmonized with it.

We may call for shortness (*c*) a *total recovery curve* and (*a*) or (*b*) a *partial recovery curve*. The total recovery curve which corresponds to unloading after the load has been left on for a very long time (when the equilibrium value, if it exists, has probably been reached) does agree with the loading curve, and the initial recovery is then equal to the initial strain, or S'_0 = total creep, very nearly. This agrees with hypothesis (i.), but contradicts hypothesis (ii.). On the other hand, the *partial* recovery curves certainly do not agree with the loading curve, and this contradicts, as we have seen, the results deduced from hypothesis (i.).

It is clear, therefore, that neither hypothesis is adequate and that the law of recovery is connected with the time during which the load has been kept on in a manner which is still obscure. Experiments on this point are still in progress.

§ 17. *Retardation-Time Curves.*

Assuming the correctness of the stress-optical law

$$r = \alpha T + \beta S$$

the retardation-time curves should be fitted by the formula

$$r = r_0 + pt^{1/3} + qt \quad \ldots \ldots \ldots \quad (34)$$

where

$$\left.\begin{aligned} r_0 &= \alpha T + \beta S_0, \\ p &= \beta . a, \\ q &= \beta . b. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (35)$$

The formula (34) was found to give quite a good fit, as indeed was only to be expected.

The above refers to the retardation-time curves on loading. The curves on unloading show the same characteristics as the strain-time recovery curves, but as a discussion of the former is necessarily dependent on a complete analysis of the latter, this must for the present be postponed.

The values of r_0/T, p/T^2, q, for O_{12}, O_{13}, O_{14} and O_{15} are shown in Table XI.

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TABLE XI.

Specimen.	Stress in bars.	r_0/T (brewsters).	p/T².	q.
O_{12}	135·4	11·669	0·00275	−0·5924
	201·1	11·938	0·00266	+1·7756
	225·2	10·451*	0·00146*	+0·4384*
O_{13}	134·4	11·563	0·00228	+0·0883
	167·5	11·582*	0·00139*	+0·0334*
	200·1	11·592	0·00304	+2·1880
O_{14}	225·2	10·252*	0·00136*	−0·2941*
O_{15}	224·2	10·080*†	0·00127*†	+0·0743*†

* Observations taken after several months' interval.

† Specimen with "skin" removed.

On the whole the values of r_0/T have remained fairly constant, but lapse of time seems to have diminished the initial stress-optical coefficient in every case except that of O_{13}. On the other hand, p/T^2 which controls the optical *creep*, shows in every case, without exception, a very serious diminution after an interval of six months. This diminution is of the order of 50 per cent. and cannot be accidental.

It appears, therefore, that some progressive change takes place in xylonite with time, which gradually diminishes the optical creep under a given load.

The coefficients q are too irregular (and really uncertain) for any useful conclusions to be drawn in their case.

The change in the stress-optical effect on removing the "skin" is very slight. The indications are that it slightly reduces the stress-optical effect, but the differences may not be significant.

Very important conclusions follow from these retardation-time curves as regards the accuracy of the estimations of stress by means of observations of the double refraction in celluloid, such as those carried out in recent years by Prof. COKER.*

The deduction of the mean stress in a specimen from the retardation observed must be seriously affected by the time which has elapsed since the load was put on. Thus, taking $p/\mathrm{T}^2 = 0\cdot0025$ as a fair average value for the earlier observations, we find the optical creep after t minutes $= (p/r_0)t^{1/3}$ of the initial retardation, approximately. Taking $r_0/\mathrm{T} = 11$ this gives a proportional error equal to $(0\cdot0025/11)\,\mathrm{T}.\,t^{1/3} = (0\cdot00023)\,\mathrm{T}.\,t^{1/3}$. Thus after 8 minutes, under a load of 200 bars (2,900 lbs. weight per square inch) the

* E. G. COKER, "The Determination, by Photo-Elastic Methods, of the Distribution of Stress in Plates of Variable Section, with some Applications to Ships' Plating" ('Trans. Inst. of Nav. Arch.,' 1911). See also, by the same author, "The Optical Determination of Stress" ('Phil. Mag.,' 1910). Also, E. G. COKER and L. N. G. FILON, "Experimental Determination of Stress and Strain in Solids" ('B. A. Report,' 1914, pp. 201 *et seq.*), and E. G. COKER, K. C. CHAKKO and M. S. AHMED, "Contact Pressures and Stresses" ('Trans. of the Inst. of Mech. Eng.,' 1921).

proportional error would amount to 9·2 per cent. After 27 minutes, or nearly half an hour, it would amount to 13·8 per cent. If we take the average value of p/T^2 given by the later observations, these creep errors will be roughly halved.

On the other hand, it is unsafe to wait until the equilibrium value is attained (even if we assume that this is ever actually reached), as the retardation is then no longer proportional to the mean stress. The observations should be taken as soon as possible after loading, and the percentage error can be diminished by reducing T, *i.e.* working with as small a load as possible.

If the stress is measured by a null method with a comparison test-piece of similar material, it is important to ensure that the test-piece and the specimen shall have been loaded *simultaneously*, or else that the time has been so long that the equilibrium value has been attained. Even then it is difficult to make sure that the specimen and test-piece are really optically similar. We have found serious divergences of α and β in specimens cut from the same sheet.

Altogether, however, the safest method should be to use as small a load as possible; observations of the isoclinic lines, that is, of the directions of principal stress, will usually allow this requirement to be fulfilled. Further, it is clearly of great importance to use for such determinations only specimens which have been kept for several years. The effect of lapse of time appears to be very great, and it is probable that with increasing age, the creep may become almost eliminated.

In this connection, Prof. COKER, to whom we have communicated privately the above results, has pointed out to us that he himself habitually uses exclusively a stock of pre-war xylonite, which he states is of much higher quality than the fresh supplies at present obtainable. Undoubtedly, after seven or eight years, great improvement in the matter of creep is to be looked for, and it seems very unlikely that any of Prof. COKER'S own results can be seriously affected by this cause. Nevertheless, as this method of investigating stresses in structures is likely to be more generally taken up, it may not be unnecessary to warn investigators, who may not have seasoned specimens of xylonite at their disposal, of this serious source of error, so that they may take the needful precautions.

§ 18. *Summary of Conclusions.*

Our conclusions may be summarized as follows :—

(1) In a thoroughly homogeneous solid, such as glass, the stress-optical effect under simple pressure shows no trace of residual effect whatever at ordinary temperatures. It attains its final value immediately the load is put on, and recovers instantly even under very high loads kept on for a very long time.

(2) In a substance such as nitro-cellulose under simple tension, on the other hand, the optical retardation shows an immediate effect on loading, very nearly proportional to the load, followed by a progressive creep until an equilibrium value is approached.

On unloading there is at first an immediate recovery, but this is only partial. A very definite residual retardation is observed, which gradually disappears completely, if enough time is allowed, at any rate for the range of loading employed which was well inside the yield point.

(3) This optical creep is comparatively large and may easily amount to 10 per cent. of the initial value in a few minutes. This percentage, however, diminishes as the load is decreased. These facts have an important bearing on stress determinations from observations of artificial double refractions in celluloid.

(4) The stretch of the celluloid follows laws closely similar to those of the optical retardation, the initial stretch being also approximately proportional to the applied load.

(5) The retardation r and stretch S are connected in each case by a linear relation

$$r = \alpha\mathrm{T}+\beta\mathrm{S},$$

T being the applied load in each case.

The constants α, β are found to vary within fairly narrow limits for any given specimen at a given time. During recovery after unloading,

$$r = \beta\mathrm{S}$$

gives a fair fit.

(6) The above linear relation is satisfactorily explained by assuming for the celluloid a heterogeneous structure, each constituent separately obeying a stress-optical law in which the retardation is rigorously proportional to the stress.

(7) The strain S on loading is connected with the time t by a formula

$$\mathrm{S} = \mathrm{S}_0+at^{1/3}+bt$$

and the stress-optical retardation r by a similar formula

$$r = r_0+pt^{1/3}+qt.$$

The constants S_0, r_0 are proportional to T ; a and p are (very roughly) proportional to T^2 ; b and q appear to vary somewhat irregularly, but are small.

(8) The exponential law, such as would follow if we assume the stress in the material to be given by the addition of elastic and viscous parts, obeying HOOKE's and STOKES' laws respectively, is definitely ruled out.

(9) The initial rate of increase, both of stretch and of retardation, is in every case very great, and the best fitting formulæ make it mathematically infinite.

(10) If the load is taken off after many hours, so that the specimen has had time to settle down, both strain and retardation on recovery give a curve identical with the corresponding curve on loading reversed. But if the load is taken off at an earlier stage, both strain and retardation on recovery follow a different law, which depends upon the time during which the original load was maintained. The form of this law has not yet been determined by us.

(11) The behaviour as regards retardation and strain of a specimen subjected to a given load is found to be influenced by its past history, in particular by previous loadings to which the specimen has been subjected. This effect dies out if sufficient time is allowed to elapse between successive loadings.

(12) The lapse of time appears to affect the strain and optical creeps in celluloid to a very considerable extent. On repeating the experiments after about six months, the constants a and p, which measure the bulk of the creep, at any rate in the earlier stages, were found to be practically halved. Thus the importance of using old and well-seasoned specimens in stress investigations cannot be too much emphasized.

In conclusion, the authors wish to acknowledge the generous help of the Department of Scientific and Industrial Research, which has enabled them to carry out the above investigation; also the valuable assistance of Mrs. J. Treweek Hughes in the taking of time observations, and of Mr. A. J. Chapman, of the Applied Mathematics Department, University College, London, by whom all the apparatus employed was made.

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